Problem 10

(a) Suppose that the dog in Problem 9 runs twice as fast as the rabbit. Find a differential equation for the path of the dog. Then solve it to find the point where the dog catches the rabbit.

(b) Suppose the dog runs half as fast as the rabbit. How close does the dog get to the rabbit? What are their positions when they are closest?

Solution

Part (a)

If the dog runs twice as fast as the rabbit, then the dog's arc length is twice as long as the rabbit's path. That is,

$$2y_{\text{rabbit}} = \text{dog's arc length from } x \text{ to } L = s|_x^L$$

The equation for the slope is then

$$\begin{aligned}
\frac{dy}{dx} &= \frac{y - \frac{1}{2}\, s|_x^L}{x - 0} \\
x\frac{dy}{dx} &= y - \frac{1}{2}\, s|_x^L \\
x\frac{dy}{dx} &= y - \frac{1}{2}\int_x^L \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx
\end{aligned}$$

Now differentiate both sides to eliminate the integral.

$$\begin{aligned}
\frac{d}{dx}\left(x\frac{dy}{dx}\right) &= \frac{d}{dx}\left[y - \frac{1}{2}\int_x^L \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx\right] \\
\cancel{\frac{dy}{dx}} + x\frac{d^2y}{dx^2} &= \cancel{\frac{dy}{dx}} - \frac{1}{2}\frac{d}{dx}\int_x^L \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
x\frac{d^2y}{dx^2} &= -\frac{1}{2}\frac{d}{dx}\int_x^L \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
x\frac{d^2y}{dx^2} &= \frac{1}{2}\frac{d}{dx}\int_L^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
x\frac{d^2y}{dx^2} &= \frac{1}{2}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}
\end{aligned}$$

Solve this equation now by using separation of variables, making use of the substitution $z = dy/dx$.

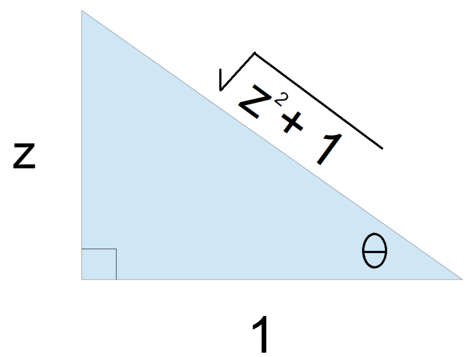
$$\begin{aligned}
x\frac{dz}{dx} &= \frac{1}{2}\sqrt{1 + z^2} \\
\frac{dz}{1 + z^2} &= \frac{1}{2}\frac{dx}{x}
\end{aligned}$$

To integrate the left side, we will use a trigonometric substitution.

$$z = \tan\theta \quad \rightarrow \quad 1 + z^2 = \sec^2\theta$$
$$dz = \sec^2\theta\, d\theta$$

$$\int \frac{\sec^2\theta\, d\theta}{\sec\theta} = \frac{1}{2}\ln x + C$$
$$\int \sec\theta\, d\theta = \frac{1}{2}\ln x + C$$
$$\ln|\sec\theta + \tan\theta| = \frac{1}{2}\ln x + C$$

To determine $\sec\theta$ and $\tan\theta$ in terms of z, draw the right triangle that is defined by the initial trigonometric substitution, $z = \tan\theta$.



$$\sec\theta = \sqrt{z^2+1} \quad \text{and} \quad \tan\theta = z.$$

So

$$\ln\left|\sqrt{z^2+1} + z\right| = \frac{1}{2}\ln x + C$$
$$e^{\ln\left|\sqrt{z^2+1}+z\right|} = e^{\ln x^{1/2} + C}$$
$$\left|\sqrt{z^2+1} + z\right| = e^C e^{\ln x^{1/2}}$$
$$\sqrt{z^2+1} + z = \pm e^C x^{1/2}$$
$$\sqrt{z^2+1} + z = Ax^{1/2}$$
$$\sqrt{z^2+1} = Ax^{1/2} - z$$
$$\cancel{z^2} + 1 = A^2 x - 2Ax^{1/2} z + \cancel{z^2}$$
$$z = \frac{A^2 x - 1}{2A\sqrt{x}}.$$

Now that we solved for z, we can solve for y by simply integrating.

$$\frac{dy}{dx} = \frac{A^2x - 1}{2A\sqrt{x}}$$
$$y = \int \frac{A^2x - 1}{2A\sqrt{x}}\,dx$$
$$y(x) = \frac{\sqrt{x}(A^2x - 3)}{3A} + D$$

All that's left is to determine the constants of integration. We use the given boundary conditions, $y(L) = 0$ and $y'(L) = 0$.

$$y(L) = D + \frac{\sqrt{L}(A^2L - 3)}{3A} = 0$$
$$y'(L) = \frac{A\sqrt{L}}{3} + \frac{A^2L - 3}{6A\sqrt{L}} = 0$$

From these equations we find that

$$A = \frac{1}{\sqrt{L}} \quad \text{and} \quad D = \frac{2L}{3}.$$

Therefore,

$$y(x) = \frac{2L}{3} + \frac{\sqrt{Lx}}{3}\left(\frac{x}{L} - 3\right).$$

The dog intercepts the rabbit when $x = 0$, i.e. when $y = 2L/3$. Therefore, the point where the dog catches the rabbit is $\left(0, \frac{2L}{3}\right)$.

Part (b)

If the dog runs half as fast as the rabbit, then the arc length of the dog's path is half as long as the rabbit's path. That is,

$$y_{\text{rabbit}} = 2(\text{dog's arc length from } x \text{ to } L) = 2\ s|_x^L$$

The equation for the slope is then

$$\frac{dy}{dx} = \frac{y - 2\ s|_x^L}{x - 0}$$
$$x\frac{dy}{dx} = y - 2\ s|_x^L$$
$$x\frac{dy}{dx} = y - 2\int_x^L \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Now differentiate both sides to eliminate the integral.

$$\begin{aligned}
\frac{d}{dx}\left(x\frac{dy}{dx}\right) &= \frac{d}{dx}\left[y - 2\int_x^L \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx\right] \\
\cancel{\frac{dy}{dx}} + x\frac{d^2y}{dx^2} &= \cancel{\frac{dy}{dx}} - 2\frac{d}{dx}\int_x^L \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\
x\frac{d^2y}{dx^2} &= -2\frac{d}{dx}\int_x^L \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\
x\frac{d^2y}{dx^2} &= 2\frac{d}{dx}\int_L^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\
x\frac{d^2y}{dx^2} &= 2\sqrt{1+\left(\frac{dy}{dx}\right)^2}
\end{aligned}$$

Solve this equation now by using separation of variables, making use of the substitution $z = dy/dx$.

$$\begin{aligned}
x\frac{dz}{dx} &= 2\sqrt{1+z^2} \\
\frac{dz}{1+z^2} &= 2\frac{dx}{x}
\end{aligned}$$

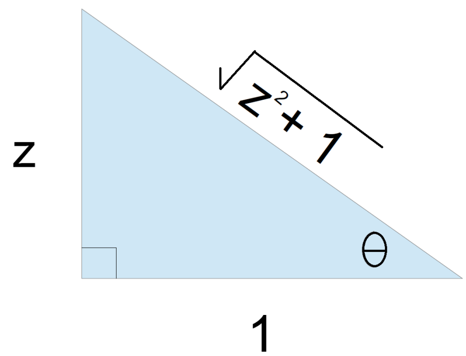
To integrate the left side, we will use a trigonometric substitution.

$$\begin{aligned}
z &= \tan\theta \quad \rightarrow \quad 1+z^2 = \sec^2\theta \\
dz &= \sec^2\theta\,d\theta
\end{aligned}$$

$$\begin{aligned}
\int \frac{\sec^2\theta\,d\theta}{\sec\theta} &= 2\ln x + C \\
\int \sec\theta\,d\theta &= 2\ln x + C \\
\ln|\sec\theta + \tan\theta| &= 2\ln x + C
\end{aligned}$$

To determine $\sec\theta$ and $\tan\theta$ in terms of z, draw the right triangle that is defined by the initial trigonometric substitution, $z = \tan\theta$.

$$\sec\theta = \sqrt{z^2+1} \quad \text{and} \quad \tan\theta = z.$$



So

$$\ln\left|\sqrt{z^2+1}+z\right| = 2\ln x + C$$
$$e^{\ln\left|\sqrt{z^2+1}+z\right|} = e^{\ln x^2 + C}$$
$$\left|\sqrt{z^2+1}+z\right| = e^C e^{\ln x^2}$$
$$\sqrt{z^2+1}+z = \pm e^C x^2$$
$$\sqrt{z^2+1}+z = Ax^2$$
$$\sqrt{z^2+1} = Ax^2 - z$$
$$\cancel{z^2} + 1 = A^2x^4 - 2Ax^2z + \cancel{z^2}$$
$$z = \frac{A^2x^4-1}{2Ax^2}$$

Now that we solved for z, we can solve for y by simply integrating.

$$\frac{dy}{dx} = \frac{A^2x^4-1}{2Ax^2}$$
$$y = \int \frac{A^2x^4-1}{2Ax^2}\,dx$$
$$y(x) = \frac{1}{2Ax} + \frac{Ax^3}{6} + D$$

All that's left is to determine the constants of integration. We use the given boundary conditions, $y(L) = 0$ and $y'(L) = 0$.

$$y(L) = D + \frac{1}{2AL} + \frac{AL^3}{6} = 0$$
$$y'(L) = -\frac{1}{2AL^2} + \frac{AL^2}{2} = 0$$

From these equations we find that

$$A = \frac{1}{L^2} \quad \text{and} \quad D = -\frac{2L}{3}.$$

Therefore,

$$y(x) = -\frac{2L}{3} + \frac{L^2}{2x} + \frac{x^3}{6L^2}.$$

In order to find out how close the dog gets to the rabbit, we must come up with an expression for the distance between them and minimize it. If we consider the equation for the tangent line to the dog's path, the distance can be expressed as the arc length of the tangent line from 0 to the point it touches the dog's path. Let $x = c$ be where the tangent line touches the dog's path. The equation of the tangent line at this point is

$$\begin{aligned} y_T - y(c) &= y'(c)(x - c) \\ y_T &= y'(c)(x - c) + y(c). \end{aligned}$$

The arc length from 0 to c represents the distance between the rabbit and the dog.

$$\int_0^c \sqrt{1 + (y_T')^2}\, dx$$

y_T' is just $y'(c)$. So we must find the value of c that minimizes

$$\int_0^c \sqrt{1 + [y'(c)]^2}\, dx. \tag{1}$$

The integrand is constant in x, so it evaluates to

$$c\sqrt{1 + \left(\frac{c^2}{2L^2} - \frac{L^2}{2c^2}\right)^2}.$$

To minimize this function, take the derivative with respect to c, set it equal to zero, and solve for c. Taking the derivative of this expression and simplifying yields

$$\frac{3c^4 - L^4}{2c^2L^2}.$$

Setting this equal to zero implies that

$$3c^4 - L^4 = 0 \quad \rightarrow \quad c = \frac{L}{3^{1/4}} \approx 0.760L.$$

Plugging this value of c into (1) yields

$$\int_0^{\frac{L}{3^{1/4}}} \sqrt{1 + \left[y'\left(\frac{L}{3^{1/4}}\right)\right]^2}\, dx = \frac{2L}{3^{3/4}} \approx 0.877L.$$

The rabbit and dog are therefore $\frac{2L}{3^{3/4}}$ units apart when they are closest. Note that we would get this same result by plugging this value of c into the distance formula,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

One point is the dog's position, $(c, y(c))$, and the other point is the rabbit's position, $(0, y_{\text{rabbit}})$. Recall, though, that y_{rabbit} is twice the dog's arc length from c to L. So for the distance formula, we have

$$\begin{aligned} d &= \sqrt{(0 - c)^2 + \left[2\int_c^L \sqrt{1 + [y'(x)]^2}\, dx - y(c)\right]^2} \\ d &= \sqrt{\left(\frac{L}{3^{1/4}}\right)^2 + \left[2\int_{\frac{L}{3^{1/4}}}^L \sqrt{1 + [y'(x)]^2}\, dx - y\left(\frac{L}{3^{1/4}}\right)\right]^2} \\ d &= \frac{2L}{3^{3/4}}. \end{aligned}$$

This verifies the result. The last part of the question can now be answered.

$$\text{Dog's Position:} \quad \left(\frac{L}{3^{1/4}}, y\left(\frac{L}{3^{1/4}}\right)\right) = \left(\frac{L}{3^{1/4}}, \frac{L}{9}\left(5\times 3^{1/4} - 6\right)\right) \approx (0.760L, 0.0649L)$$

$$\text{Rabbit's Position:} \quad \left(0, 2\int_{\frac{L}{3^{1/4}}}^{L} \sqrt{1+[y'(x)]^2}\,dx\right) = \left(0, \frac{2L}{9}\left(4\times 3^{1/4} - 3\right)\right) \approx (0, 0.503L)$$

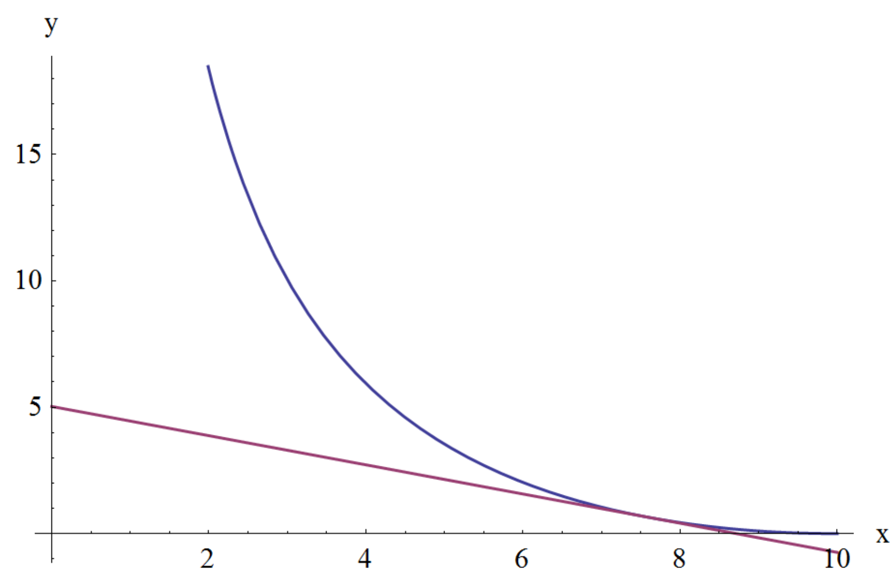


Figure 1: Plot of the dog's path and tangent line at closest point to rabbit for $L = 10$.

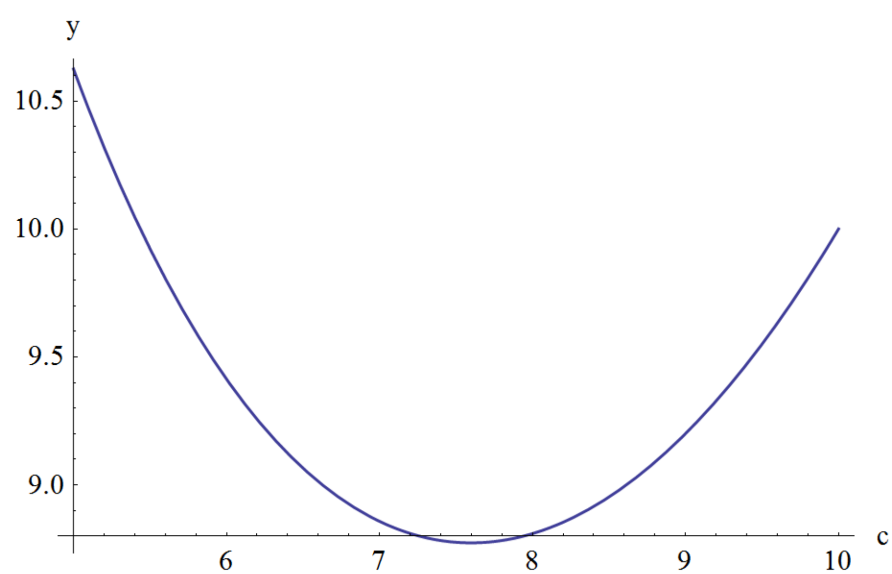


Figure 2: Plot of distance from dog to rabbit (1) for $L = 10$. Clearly it is minimum when $c = \frac{L}{3^{1/4}}$.